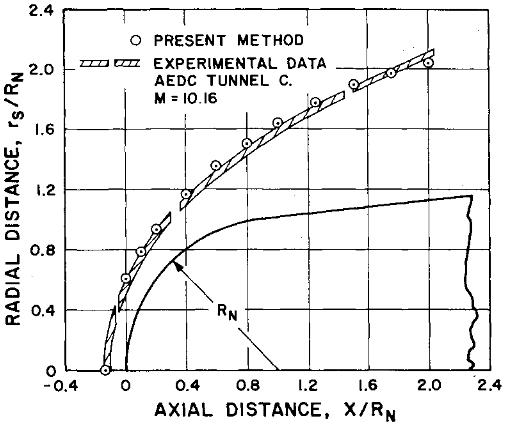

Fig. 5 Comparison of theoretical and experimental shock shape, nose region.

sented in Fig. 1. In order to facilitate the use of these correlations for specific flight conditions, the density ratio parameters calculated for various Mach number and altitude combinations are presented in Fig. 2. This plot is for an inviscid gas at chemical equilibrium.[6,7]

By use of flow-field data, Eq. (2) has been solved for the eccentricity factor e_s. These results (shown in Fig. 3) indicate a dependence upon the density ratio as expected; however, a dependence upon the axial distance is also indicated. It is further noted that, for any given axial station, e_s is nonlinear with density ratio and peaks at approximately $\rho_\infty/\rho_2 = 0.095$.

Equation (2) may now be solved for almost any combination of Mach number and altitude of interest ($0.05 \leq \rho_\infty/\rho_2 \leq 0.25$) to obtain r_s as a function of X. Figures 1–3 provide all of the necessary data.

The results of the application of this method are presented in Fig. 4 for several freestream conditions and are compared with General Electric Missile and Space Division flow-field data. Figure 5 presents theoretical and experimental shock-shape data. These comparisons indicate excellent agreement between the correlation method [Eq. (2) with Figs. 1–3] and the exact flow-field data.

Approximate Method

A more approximate solution for the bow shock wave has also been obtained by approximating the eccentricity data in Fig. 3 with a single analytical expression, determining relations for the curves in Fig. 1 and incorporating these expressions into Eq. (2). This approximate solution may be expressed by the single equation

$$\frac{r_s}{R_N} = \left\{4.18\left(\frac{\rho_\infty}{\rho_2}\right)^{0.1958}\left[\frac{X}{R_N} + 0.880\left(\frac{\rho_\infty}{\rho_2}\right)^{1.053}\right] - 0.646\left[\frac{X}{R_N} + 0.880\left(\frac{\rho_\infty}{\rho_2}\right)^{1.053}\right]^{1.467}\right\}^{0.5} \quad (3)$$

The results of the application of this approximate relation [Eq. (3)] are also presented in Fig. 4 where relatively good agreement is indicated.

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Natural Frequencies of Orthotropic Circular Plates

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This note is concerned with the determination of natural frequencies of orthotropic circular plates. A circular plate fabricated with radial and/or circumferential reinforcements may be idealized as an orthotropic plate of uniform thickness with different material properties in the radial and circumferential directions. The stress-strain relations for such plates can be written in a manner analogous to that given in Ref. 1:

$$\begin{aligned}\sigma_r &= E_1\epsilon_r + E_{12}\epsilon_\theta \\ \sigma_\theta &= E_2\epsilon_\theta + E_{12}\epsilon_r \\ \tau_{r\theta} &= G\gamma_{r\theta}\end{aligned} \quad (1)$$

where E_1, E_{12}, E_2, and G are material constants. With the usual assumptions of small-deflection theory of plates, the following relations between moments and the lateral deflection $W(r,\theta)$ of the plate are obtained:

$$\begin{aligned}M_r &= -D_r\left[\frac{\partial^2 W}{\partial r^2} + \alpha\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right)\right] \\ M_\theta &= -D_r\left[\beta\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial\theta^2}\right) + \alpha\frac{\partial^2 W}{\partial r^2}\right] \\ M_{r\theta} &= -2\gamma D_r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial W}{\partial\theta}\right]\end{aligned} \quad (2)$$

In the foregoing, $D_r = E_1h^3/12$, $\alpha = E_{12}/E_1$, $\beta = E_2/E_1$, and $\gamma = G/E_1$. Also, h is the thickness of the plate. The governing equilibrium equation for the plate in terms of these moments is

$$\frac{\partial^2}{\partial r^2}(rM_r) + \frac{2}{r}\frac{\partial}{\partial r}\left(r\frac{\partial M_{r\theta}}{\partial\theta}\right) - \frac{\partial M_\theta}{\partial r} + \frac{1}{r}\frac{\partial^2 M_\theta}{\partial\theta^2} = -rq(r,\theta) \quad (3)$$

where $q(r,\theta)$ is the load intensity. For a free vibration problem, $q = -\rho(\partial^2 W/\partial t^2)$, where $W = W(r,\theta,t)$ and ρ is mass per unit area of the plate. Substitution of Eq. (2) into Eq. (3) yields the governing equation for the free vibration of the plate as

$$r\frac{\partial^4 W}{\partial r^4} + 2\frac{\partial^2 W}{\partial r^3} - \beta\left(\frac{1}{r}\frac{\partial^2 W}{\partial r^2} - \frac{1}{r^2}\frac{\partial W}{\partial r}\right) + \frac{\beta}{r^3}\left(\frac{\partial^4 W}{\partial\theta^4} + 2\frac{\partial^2 W}{\partial\theta^2}\right) + 2(\alpha + 2\gamma) \times \frac{\partial^2}{\partial\theta^2}\left(\frac{1}{r}\frac{\partial^2 W}{\partial r^2} - \frac{1}{r^2}\frac{\partial W}{\partial r} + \frac{1}{r^3}W\right) = -\rho r\frac{\partial^2 W}{\partial t^2} \quad (4)$$

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If the new variable $x = kr$ with $k^4 = \rho\omega^2/D_r$ is introduced and W assumed as

$$W\alpha\left[R_m(X)\begin{matrix}\cos\\ \sin\end{matrix} m\theta\right]e^{i\omega t} = \left\{\left[\sum_{n=0}^{\infty} A_{m,n}x^{m+n}\right]\begin{matrix}\cos\\ \sin\end{matrix} m\theta\right\}e^{i\omega t} \tag{5}$$

Eq. (4) yields the following recurrence relation for the coefficients $A_{m,n}$:

$$A_{m,n} = \frac{A_{m,(n-4)}}{(m+n)(m+n-2)[(m+n-1)^2-\beta] + m^2[(m^2-2)\beta - 2(\alpha+2\gamma)(m+n-1)^2]} \tag{6}$$

The axisymmetric modes corresponds to $m = 0$. In view of the recurrence relations (6), the function R_m can be written in terms of the first four coefficients, namely, $A_{m,0}$, $A_{m,1}$, $A_{m,2}$, and $A_{m,3}$. At the center of the plate ($r = 0$) the two conditions to be enforced are that for all values of m: 1) the left-hand side of Eq. (4) must vanish, and 2) the moments must be finite. These conditions lead to $A_{m,1} = A_{m,3} = 0$ so that the function $R_m(X)$ becomes

$$R_m = A_{m,0}\sum_{j=0,4,8}^{\infty} C_{m,j}X^{m+j} + A_{m,2}\sum_{j=0,4,8}^{\infty} D_{m,j}X^{m+j+2} \tag{7}$$

where $C_{m,j} = A_{m,j}/A_{m,0}$ and $D_{m,j+2} = A_{m,j+2}/A_{m,2}$. Equation (6), when specialized for the isotropic case [$\alpha = \nu =$ Poisson's ratio, $\beta = 1$ and $(\alpha + 2\gamma) = 1$], reduces to

$$A_{m,n} = A_{m,(n-4)}/n(n-2)(2m+n)(2m+n-2) \tag{8}$$

With the coefficients given by Eq. (8), the series for R_m [Eq. (7)] can be shown to be a linear combination of the Bessel functions $J_m(X)$ and $I_m(X)$. Therefore, for the isotropic plate, the solution reduces to that given by Rayleigh[2] and by Morse.[3]

When the boundary conditions $W = \partial W/\partial r = 0$ at the edge $r = a$ ($X = ka = X_1$) for a clamped plate or $W = M_r = 0$ at $r = a$ for a simply supported plate are enforced, Eq. (7) yields the following characteristic equations for the determination of the natural frequencies:

$$\left[\sum_j C_{m,j}X_1^{m+j}\right]\left[\sum_j (m+j+2)D_{m,j+2}X_1^{m+j+1}\right] - \left[\sum_j (m+j)C_{m,j}X_1^{m+j-1}\right]\left[\sum_j D_{m,j+2}X_1^{m+j+2}\right] = 0 \tag{9}$$

$$\left[\sum_j C_{m,j}X_1^{m+j}\right]\left[\sum_j \{(m+j+2)(m+j+1+\alpha) - \alpha m^2\}D_{m,j+2}X_1^{m+j}\right] - \left[\sum_j \{(m+j)(m+j-1+\alpha) - \alpha m^2\}C_{m,j}X_1^{m+j-2}\right]\left[\sum_j D_{m,j+2}X_1^{m+j+2}\right] = 0 \tag{10}$$

If the infinite series appearing in the foregoing equations is restricted to include terms up to the power $(2m + 4)$, a first approximation for the values of (ka) corresponding to the lowest frequencies are obtained as

$$(ka)^4_{\text{clamped}} = 1/(C_{m,4} - 3D_{m,6}) \tag{11}$$

$$(ka)^4_{s,s} = \frac{2m+1+\alpha}{2m+5+\alpha}(ka)^4_{\text{clamped}} \tag{12}$$

For the isotropic plate the foregoing equations simplify to

$$(ka)^4_{\text{clamped}} = (m+1)(m+2)(m+3)\cdot 2^4$$
$$(ka)^4_{ss} = \frac{2m+1+\nu}{2m+5+\nu}(ka)^4_{\text{clamped}} \tag{13}$$

Using Eqs. (13) with $\nu = \frac{1}{3}$, values of (ka) for $m = 0$, 1, and 2 are, respectively, 3.13, 4.43, and 5.57 for the clamped plate and 2.21, 3.63, and 4.84 for the simply supported plate. These values obtained as first approximations agree well with those obtained from the solution in terms of the Bessel functions.

If for $m = 0$, a second approximation involving terms up to the eighth power in (ka) is considered for the clamped plate, the values of the first two frequencies are obtained from $(ka) = 3.198$ and 5.855, whereas the solution to the characteristic equation in terms of the Bessel functions yields $(ka) = 3.2$ and $(ka) \cong 6.3$.[2] For $m = 0$, for the orthotropic clamped plate, the first approximation [Eq. (11)] gives, $(ka)^4 = (9 - \beta)(25 - \beta)/2$, from which it can be concluded that for $\beta \gtrless 1$, $(ka)^4 \lessgtr 96$.

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A Heat-Transfer Criterion for the Detection of Incipient Separation in Hypersonic Flow

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PREVIOUS studies of flow separation resulting from shock-wave boundary-layer interaction in supersonic flow have encountered the difficulty of detecting incipient separation. When the region of separation is relatively large, the flow pattern exhibits the characteristic shock wave from the separation point visible upstream of the agency provoking separation. The pressure distribution then exhibits the familiar plateau pressure occurring between separation and reattachment. The first appearance of a knee in the pressure distribution, generating three points of inflexion instead of the one typical of an attached flow, has been used by many investigators to indicate the onset of separation. The applicability of this criterion to high Mach number studies conducted in intermittent facilities with very short running times is doubtful on account of the low accuracy of pressure measurements. Either an alternative or additional criterion is desirable.

In a current research program being conducted in the Imperial College Hypersonic Gun Tunnel, the separation of the flat plate laminar boundary layer due to the interaction with either an externally generated oblique shock wave or a wedge compression-corner is being investigated at a Mach number of 10. As a precursor to the main part of the work, measurements of the heat-transfer rates in the interaction region were made using thin-film resistance thermometer gages.†

Pressure measurements obtained recently for identical configurations have substantiated the results of the heat-transfer work. In Fig. 1, the pressure- and heat-transfer distributions obtained on the wedge compression-corner model for various wedge angles are presented together with the corresponding schlieren photographs of the flow in the neighborhood of the corner. In Fig. 1a, the flow is fully attached with

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